LABORATORIO DI MATEMATICA

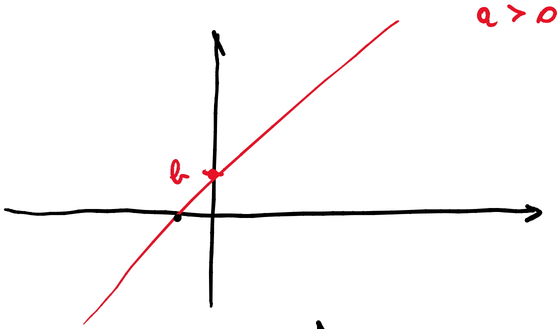
venerdì 9 febbraio 2024 09:38

Grafici di alcune funzioni elementari

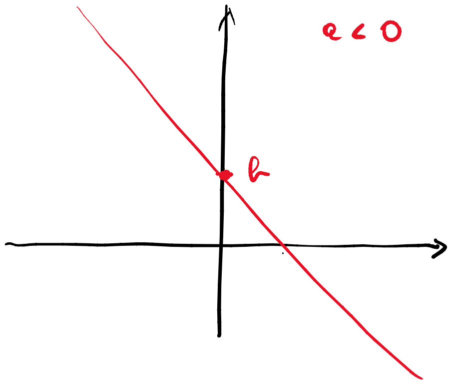
1) Polinomi di primo grado.

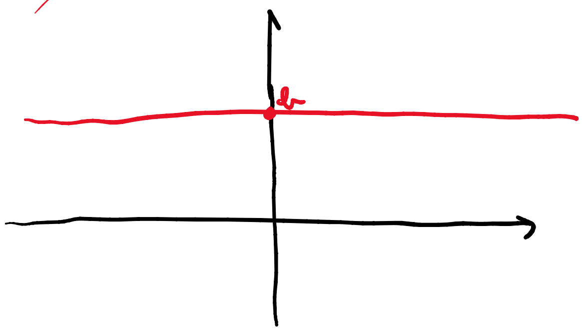
- $f(x) = ax + b$

$a > 0$



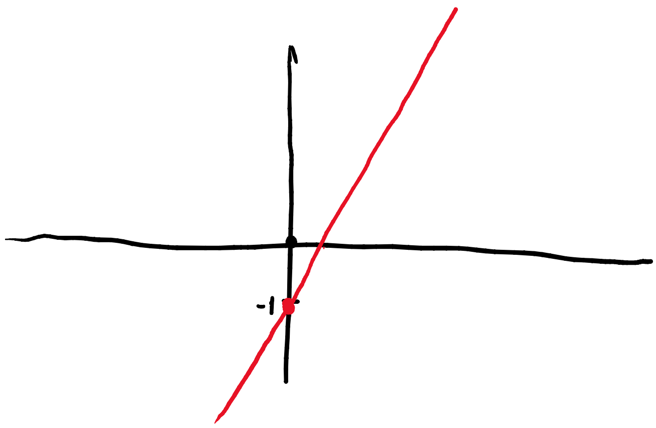
$a < 0$



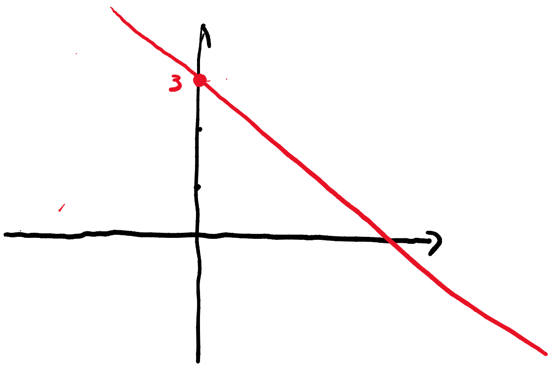


ESEMPI

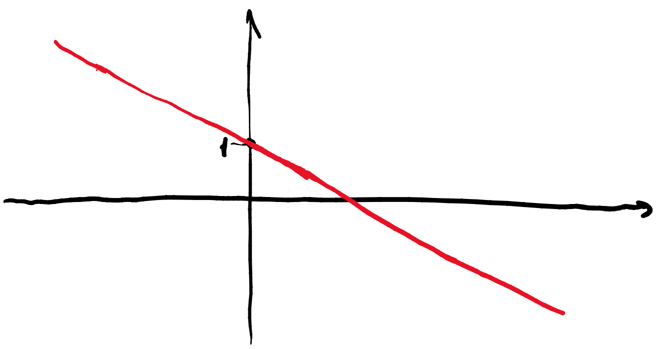
$f(x) = 2x - 1$



$f(x) = -x + 3$



$f(x) = 1 - \frac{x}{2} = -\frac{x}{2} + 1$



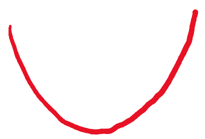
2) Polinomi di secondo grado:

$$f(x) = a x^2 + b x + c \quad (\text{con } a \neq 0)$$

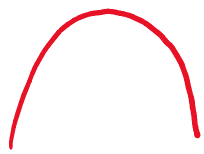
Il grafico è una parabola.

- la concavità della parabola dipende dal segno di a:

$a > 0$



$a < 0$



- Il n. di intersezioni con l'asse x dipende dal DISCRIMINANTE:

$$\Delta = b^2 - 4ac$$

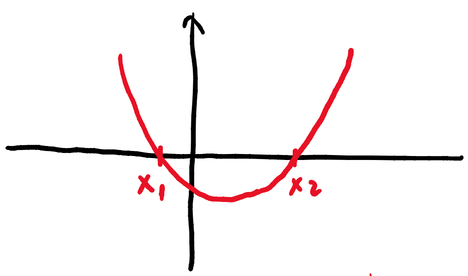
Se $\Delta > 0$ la parabola interseca l'asse x in due punti:

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

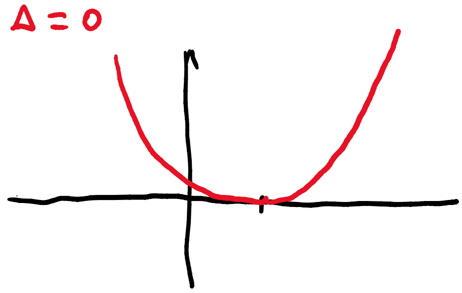
Se $\Delta = 0$, la parabola tocca l'asse in un solo punto $x = -\frac{b}{2a}$

Se $\Delta < 0$ la parabola non tocca mai l'asse delle x.

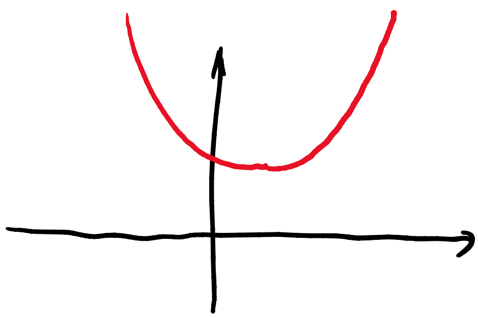
- $a > 0$
  $\Delta > 0$



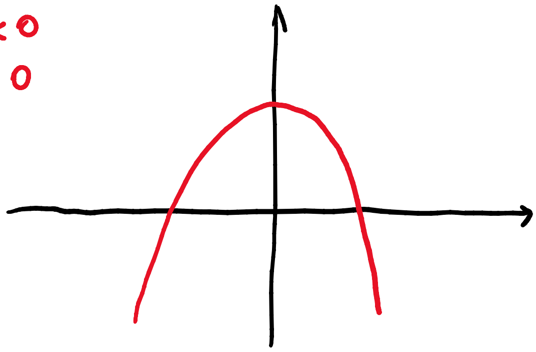
$a > 0$
$\Delta = 0$



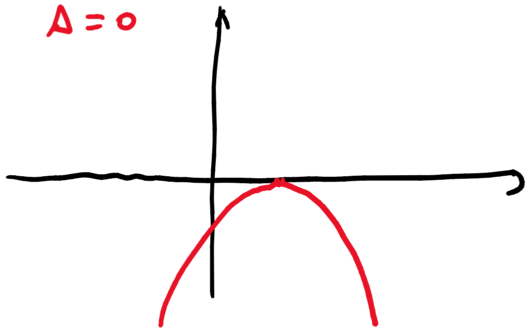
$a > 0$
$\Delta < 0$



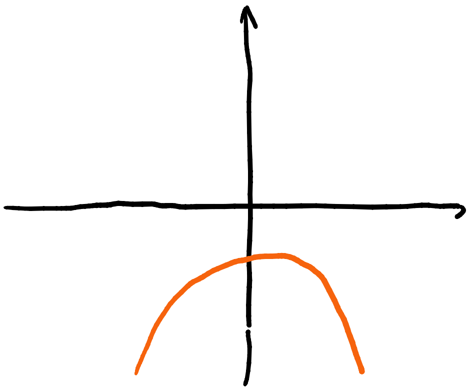
$a < 0$
$\Delta > 0$



$a < 0$
$\Delta = 0$



$a < 0$
$\Delta < 0$

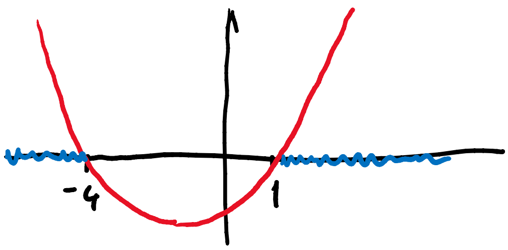


Questi grafici permettono di risolvere disequazioni di 2° grado.

- $x^2 + 3x - 4 \geq 0 \qquad (a = 1, \ b = 3, \ c = -4)$

$\Delta = 9 + 16 = 25 > 0$

$x_{1,2} = \dfrac{-3 \pm \sqrt{25}}{2} = \begin{cases} 1 \\ -4 \end{cases}$

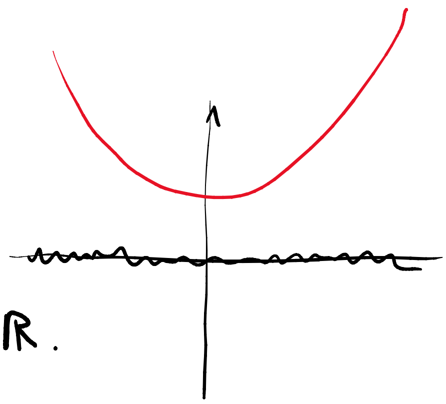


Soluzioni della disequazione:

$x \leq -4 \quad \vee \quad x \geq 1.$

- $x^2 + 3x + 4 \geq 0$

$\Delta = 9 - 16 = -7 < 0$



La disequazione è vera $\forall x \in \mathbb{R}$.

- $x^2 + 3x + 4 \leq 0$

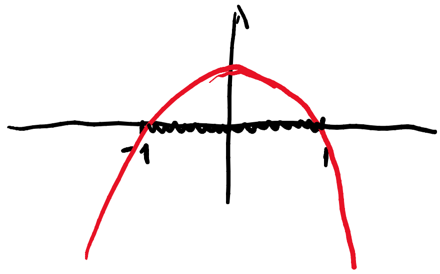
  $\nexists$ soluzioni.

- $1 - x^2 \geq 0$

  $-x^2 + 1 \geq 0$ $\quad (\Delta = 0 + 4 = 4 > 0)$

  $(-x^2 + 1 = 0 \Leftrightarrow x^2 = 1 \Leftrightarrow x = 1 \lor x = -1)$

  Le soluzioni della disequazione sono $-1 \leq x \leq 1$.



ESERCIZIO

Risolviamo

$$\frac{|x-1| - 3}{x^2 - 3x - 10} \geq 0.$$

Numeratore:

$|x-1| - 3 \geq 0$

$|x-1| \geq 3$

$x - 1 \geq 3 \quad \lor \quad x - 1 \leq -3$

$x \geq 4 \quad \lor \quad x \leq -2$

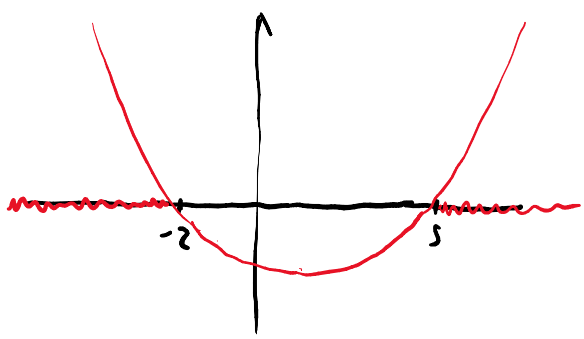
Denominatore

$x^2 - 3x - 10 \geq 0$

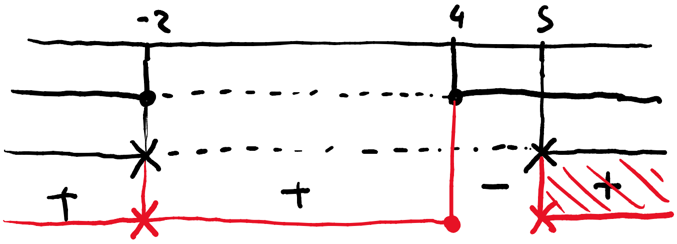
$\Delta = 9 + 40 = 49 > 0$

$x_{1,2} = \frac{3 \pm 7}{2} \begin{cases} 5 \\ -2 \end{cases}$

$x \geq 5 \quad \lor \quad x \leq -2$



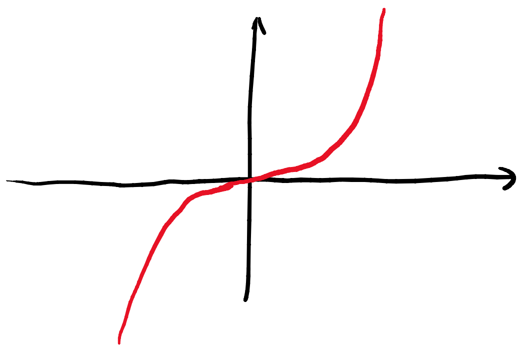
Segno della frazione:



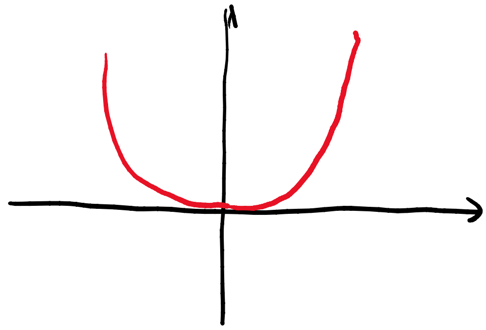
$x < -2 \quad \vee \quad -2 < x \le 4 \quad \vee \quad x > 5.$

Potenze Naturali

$f(x) = x^3$



$f(x) = x^4$

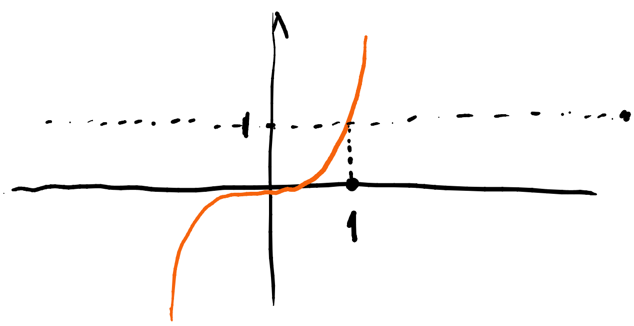


$f(x) = x^n$ ha un grafico simile a quello di x^3 se n è dispari ($n \ge 5$) e simile a quello di x^4 se n è pari ($n \ge 6$).

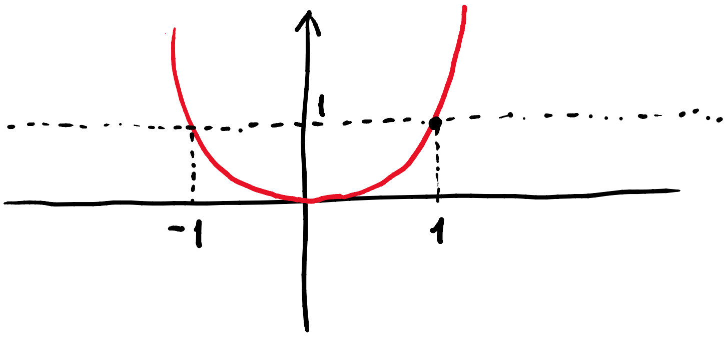
- $x^3 \le 1$

  $\sqrt[3]{x^3} \le \sqrt[3]{1}$

  $x \le 1$



- $x^4 \le 1$



Si può fare la radice quarta me

$\sqrt[4]{x^4} = |x|$

$x^4 \le 1$

$\sqrt[4]{x^4} \le 1$

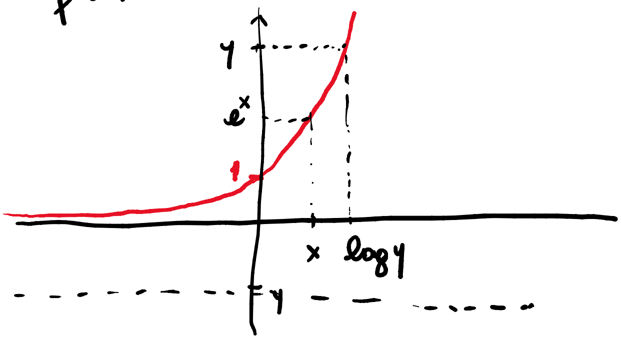
$|x| \le 1$

$-1 \le x \le 1$

Ricordare:

$\sqrt[n]{x^n} = \begin{cases} x & \text{se } n \text{ è dispari} \\ |x| & \text{se } n \text{ è pari.} \end{cases}$

$f(x) = e^x$



Se consideriamo l'equazione

$e^x = y$

Ci sono 2 casi:

- Se $y \le 0$ $e^x = y$ non ha soluzioni
- Se $y > 0$ la soluzione è di $e^x = y$ è $x = \log y$.

ESEMPI

- $e^x = 2 \Rightarrow x = \log 2$
- $e^x = -1 \Rightarrow$ non ha soluzioni.
- $e^x = e^2 \Rightarrow x = \log e^2 = 2$

Ricordare
$\log e^x = x$

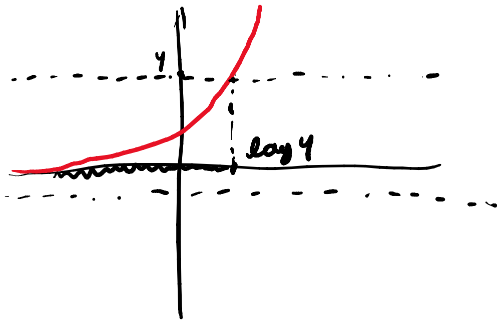
- $e^{x^2} = 3$

  $x^2 = \log 3$

  $x = \pm\sqrt{\log 3}$.

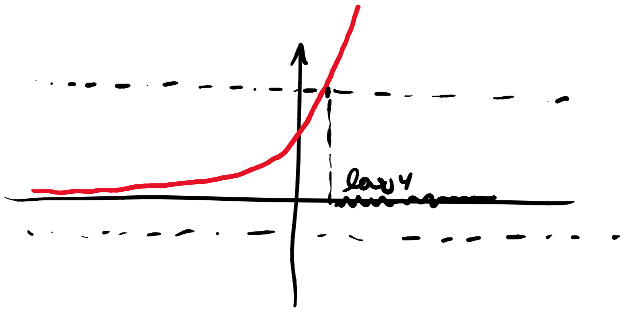
Disequazioni: $e^x \le y$

- Se $y \le 0$: $\nexists\ x \in \mathbb{R}$
- Se $y > 0$: $x \le \log y$



— $e^x \ge y$

- $y \le 0$: $e^x \ge y \quad \forall x \in \mathbb{R}$
- $y > 0$: $x \ge \log y$



ESEMPI

- $e^x \ge 4 \Rightarrow x \ge \log 4$
- $e^x + 2 \le 0 \Rightarrow e^x \le -2 \qquad \nexists\ x \in \mathbb{R}$
- $3e^x + 2 > 0 \Rightarrow 3e^x > -2 \Leftrightarrow e^x > -\frac{2}{3} \qquad \forall x \in \mathbb{R}.$

ESERCIZIO

$$\frac{|e^x - 4| - 2}{e^x - 1} \le 0$$

$t = e^x$

$$\frac{|t-4| - 2}{t - 1} \le 0$$

Numeratore: $|t-4| - 2 \ge 0$

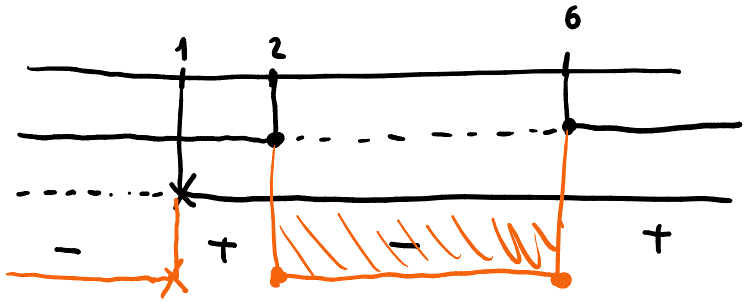
$|t-4| \ge 2$

$t - 4 \ge 2 \quad \vee \quad t - 4 \le -2$

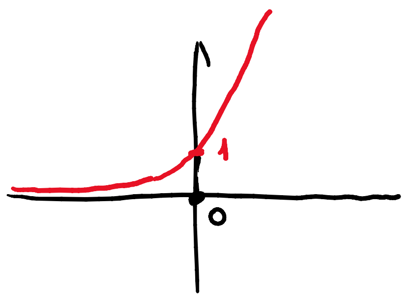
$t \ge 6 \quad \vee \quad t \le 2$

Denominatore: $t-1 \geq 0 \iff t \geq 1$.

Segno della frazione:



$t < 1 \quad \vee \quad 2 \leq t \leq 6$



$e^x < 1 \quad \vee \quad 2 < e^x < 6$

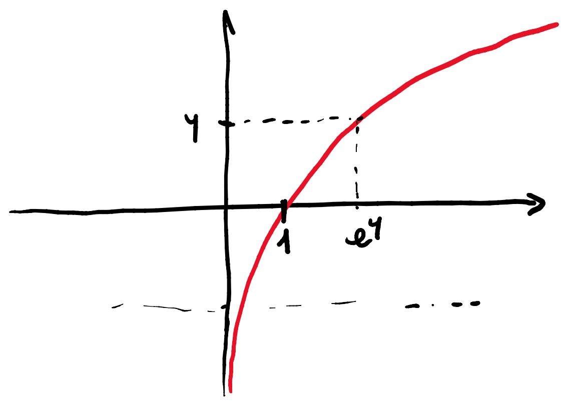
$x < \log 1 \qquad \log 2 \leq x \leq \log 6$

$x < 0$

Soluzioni: $x < 0 \quad \vee \quad \log 2 \leq x \leq \log 6$.

$f(x) = \log x$

$\log x$ è definito solo se $x > 0$.

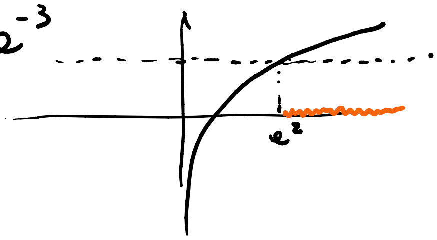


$\log x = y \Rightarrow x = e^y$.

$\log x = 3 \Rightarrow x = e^3$

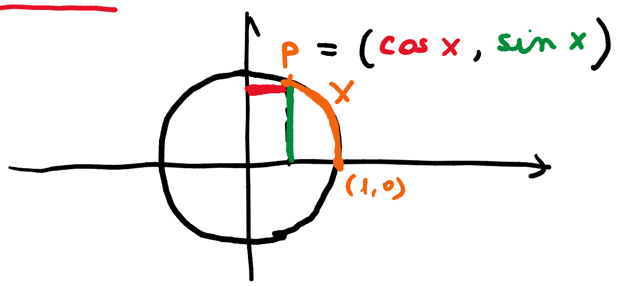
$\log x = -3 \Rightarrow x = e^{-3}$

$\log x \geq 2 \Rightarrow x \geq e^2$



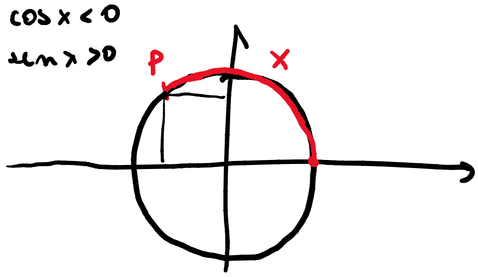
Funzioni Trigonometriche

Seno e coseno



$\cos x < 0$

$\operatorname{sen} x > 0$



$\cos 0 = 1 \qquad \sin 0 = 0$

$\cos \frac{\pi}{2} = 0 \qquad \operatorname{sen} \frac{\pi}{2} = 1$

$\cos \pi = -1 \qquad \operatorname{sen} \pi = 0$

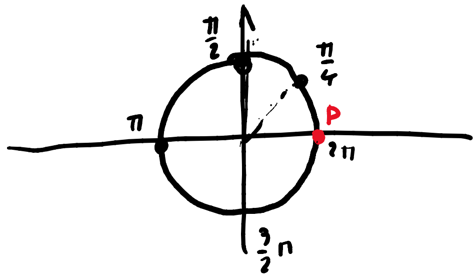
$\cos \frac{3}{2}\pi = 0 \qquad \operatorname{sen} \frac{3}{2}\pi = -1$

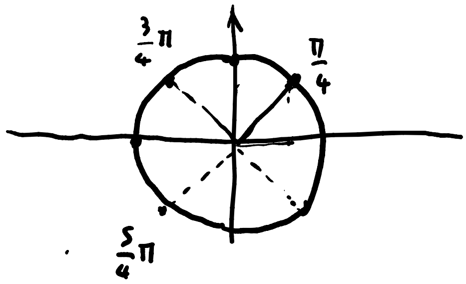
$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \operatorname{sen} \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos\left(\frac{3}{4}\pi\right) = -\frac{1}{\sqrt{2}} \qquad \operatorname{sen} \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos\left(\frac{5}{4}\pi\right) = -\frac{1}{\sqrt{2}} \qquad \operatorname{sen}\left(\frac{5}{4}\pi\right) = -\frac{1}{\sqrt{2}}$

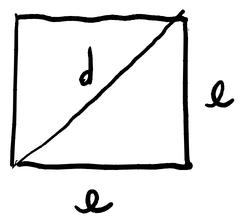
$\cos\left(\frac{7}{4}\pi\right) = \frac{1}{\sqrt{2}} \qquad \operatorname{sen}\left(\frac{7}{4}\pi\right) = -\frac{1}{\sqrt{2}}$





Ricordare

$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, infatti: $\frac{\sqrt{2}}{2} = \frac{\cancel{\sqrt{2}}}{\sqrt{2}\cdot\cancel{\sqrt{2}}} = \frac{1}{\sqrt{2}}$



$d = \sqrt{l^2 + l^2} = \sqrt{2l^2}$

$= \sqrt{2}\, l$

se $d = 1$, $l = \frac{1}{\sqrt{2}}$

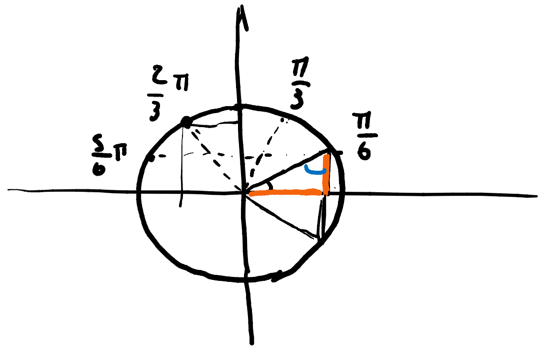
Multipli di $\frac{\pi}{6}$ ($30°$)

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ $\quad$ $\sin\frac{\pi}{6} = \frac{1}{2}$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ $\quad$ $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{2}{3}\pi\right) = -\frac{1}{2}$ $\quad$ $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{5}{6}\pi\right) = -\frac{\sqrt{3}}{2}$ $\quad$ $\sin\left(\frac{5}{6}\pi\right) = \frac{1}{2}$



Numeri complessi

- $\mathbb{C} := \{ x + iy$ con $x, y \in \mathbb{R}$ e i è un numero tale che $i^2 = -1 \}$

I numeri complessi si possono sommare / moltiplicare fra loro:

$$
\begin{aligned}
& 2+3i + (3+i)(1-i) \\
&= 2+3i + (3-3i+i-i^2) \\
&= 2+3i + (3-2i-(-1)) \\
&= 2+3i+3-2i+1 \\
&= 6+i
\end{aligned}
$$

La forma $x+iy$ di un numero complesso è detta forma algebrica. (o forma cartesiana).

ESEMPIO

Scriviamo in forma algebrica il numero: $\frac{1+i}{2+i}$

$$\frac{1+i}{2+i} = \frac{1+i}{2+i} \cdot \frac{(2-i)}{(2-i)} = \frac{(1+i)(2-i)}{(2+i)(2-i)}$$

$$= \frac{2-i+2i-i^2}{4-i^2}$$

$$= \frac{2+i+1}{4+1} = \frac{3+i}{5} = \frac{3}{5} + \frac{1}{5}i.$$

- Se $p(x)$ è un polinomio di II grado e $\Delta < 0$, $p(x) = 0$ non ha soluzioni reali.
  Ha due soluzioni complesse:

$$x^2 + x + 2 = 0$$

$$\Delta = 1 - 8 = -7 < 0$$

le soluzioni complesse sono:

$$z_{1,2} = \frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm \sqrt{7}i}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$$

$$\boxed{(\sqrt{7}\,i)^2 = (\sqrt{7})^2 i^2 = 7 i^2 = -7}$$

- $x^2 + 4 = 0$

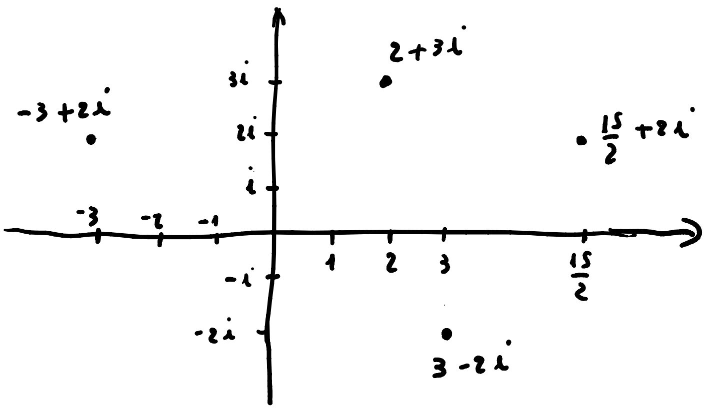
  $x^2 = -4$ non ci sono soluzioni reali.

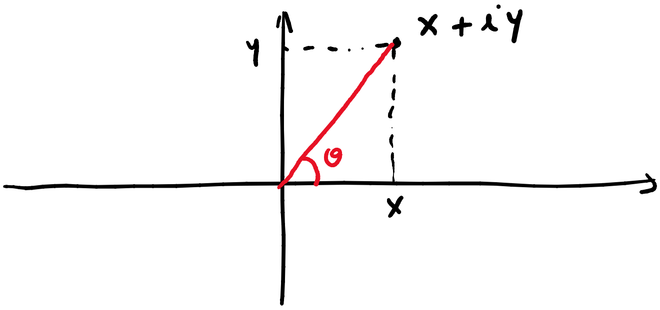
  Ma ci sono soluzioni complesse:

  $x = \pm\sqrt{-4} = \pm\sqrt{4}\,i = \pm 2i.$

Altre rappresentazioni dei numeri complessi:

- Graficamente i numeri complessi si rappresentano su un piano:





- la distanza di $x+iy$ dall'origine è detto **MODULO** del numero complesso e si calcola tramite il teorema di Pitagora:

  $|x+iy| = \sqrt{x^2+y^2}$

- l'angolo θ è l'unico angolo in $[0, 2\pi)$ tale che

  $\cos\theta = \frac{x}{|x+iy|}$ e $\sin\theta = \frac{y}{|x+iy|}$ (θ è detto **ARGOMENTO** di $x+iy$)

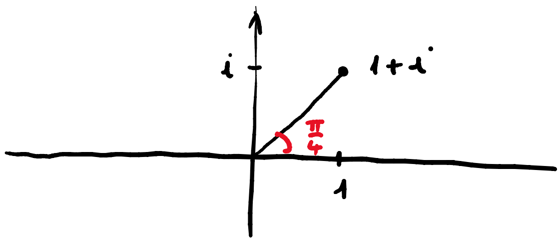
ESEMPI

$z = 1 + i$ ($z = 1+i \Rightarrow x=1, y=1$)

$|z| = \sqrt{1^2+1^2} = \sqrt{2}$

L'angolo θ soddisfa $\cos\theta = \frac{1}{\sqrt{2}}$ e $\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$

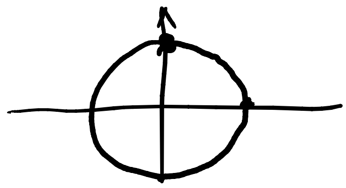


Se conosciamo φ e $|z|$ allora

$z = |z| \cos\varphi + i\, |z| \operatorname{sen}\varphi$ $\qquad \left(\begin{array}{l} x = |z|\cos\varphi \\ y = |z| \operatorname{sen}\varphi \end{array} \right)$

Inoltre si dimostra che $\forall\, n \in \mathbb{N}$:

$z^n = |z|^n \cos(n\varphi) + i\, |z|^n \sin(n\varphi)$.



ESEMPIO

Calcoliamo $(1+i)^{10}$

Allora visto che $|1+i| = \sqrt{2}$ e $\arg(1+i) = \frac{\pi}{4}$

$$(1+i)^{10} = (\sqrt{2})^{10} \cos\left(10 \cdot \tfrac{\pi}{4}\right) + i\, (\sqrt{2})^{10} \operatorname{sen}\left(10 \cdot \tfrac{\pi}{4}\right)$$

$$= 32 \underbrace{\cos\left(\tfrac{5}{2}\pi\right)}_{=0} + i\; 32 \underbrace{\sin\left(\tfrac{5}{2}\pi\right)}_{1}$$

$$= 32\, i \, .$$

- $z = \sqrt{3} + i$ $\qquad (x = \sqrt{3}, \; y = 1)$.

Calcoliamo z^5:

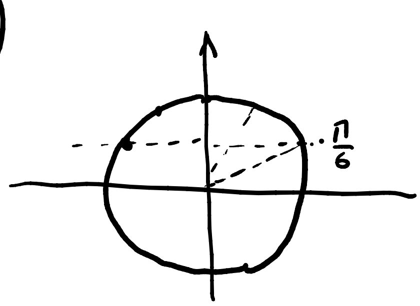
$|z| = \sqrt{3+1} = 2$

$\cos\varphi = \frac{\sqrt{3}}{2} \qquad \sin(\varphi) = \frac{1}{2} \qquad \Rightarrow \qquad \varphi = \frac{\pi}{6}$

$$z^5 = 2^5 \cos\left(5 \cdot \tfrac{\pi}{6}\right) + i\, 2^5 \sin\left(5 \cdot \tfrac{\pi}{6}\right)$$

$$= 32 \cdot \left(-\tfrac{\sqrt{3}}{2}\right) + i\; 32 \cdot \tfrac{1}{2}$$

$$= -16\sqrt{3} + 16\, i \, .$$



Radici n-esime di numeri complessi

Sia $z = |z| \cos\theta + i\, |z| \operatorname{sen}\theta$.

Vogliamo trovare le radici n-esime di z cioè i numeri complessi w tali che $w^n = z$.

$$w = |w| \cos(\alpha) + i\, |w| \sin\alpha$$

$$w^n = |w|^n \cos(n\alpha) + i\, |w|^n \sin(n\alpha)$$

Quando $w^n = z$?

$$\begin{cases} |w|^n = |z| \\ n\alpha = \theta + 2k\pi \end{cases} \qquad k = 0, 1, 2, \dots, n-1.$$

$$\begin{cases} |w| = \sqrt[n]{|z|} \\ \alpha = \dfrac{\theta}{n} + \dfrac{2k\pi}{n} \end{cases} \qquad k = 0, 1, 2, \dots, n-1$$

Formula per le radici n-esime

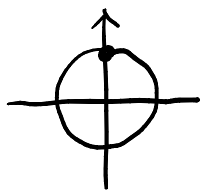
$$w = \sqrt[n]{|z|} \cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i \sqrt[n]{|z|} \operatorname{sen}\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)$$

$$k = 0, 1, 2, \dots, n-1.$$

ESEMPIO

$z = i$ Cerchiamo le radici cubiche di z.

$(x = 0,\ y = 1)$



$$|z| = \sqrt{0^2 + 1^2} = 1$$

$\arg(z)$: $\cos\theta = 0$ $\operatorname{sen}\theta = 1 \Rightarrow \arg z = \frac{\pi}{2}$

Le radici cubiche di z sono:

$$w = \sqrt[3]{1} \cos\left(\frac{1}{3}\frac{\pi}{2} + \frac{2k\pi}{3}\right) + \sqrt[3]{i} \operatorname{sin}\left(\frac{1}{3}\frac{\pi}{2} + \frac{2k\pi}{3}\right) \qquad k = 0, 1, 2.$$

Cioè:

$k=0: \quad w = \cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{i}{2}$

$n=1: \quad w = \cos\left(\frac{5}{6}\pi\right) + i \sin\left(\frac{5}{6}\pi\right) = -\frac{\sqrt{3}}{2} + \frac{i}{2}$

$k=2: \quad w = \cos\left(\frac{3}{2}\pi\right) + i \sin\left(\frac{3}{2}\pi\right) = -i$